ENS PARIS SACLAY

End-of-studies internship report

Properties of large dynamic Lotka Volterra systems equilibria for theoretical ecology

LABORATOIRE D'INFORMATIQUE GASPARD MONGE

Kayané ROBACH

kayane.robach@ens-paris-saclay.fr

Internship supervisor: **Jamal NAJIM, LIGM**

Referent teacher: **Alain TROUVÉ, ENS PARIS SACLAY**

April 18, 2022 - September 19, 2022

Contents

Acknowledgements

I would like to thank Jamal Najim, CNRS Researcher at LIGM, my internship tutor, who made me dive into the world of random matrix theory first, during the master, and then into its application to theoretical ecology throughout this research work. I thank Maxime Clenet, almost PhD, for his precious suggestions and the discussions we had. Thanks to all the interns, especially Julien, Siwan and Theo for our mathematical talks. I am grateful for the collaboration of Walid Hachem, CNRS Researcher at LIGM, his advice and his introduction on approximate message passing methods. Finally, I would like to thank all doctoral student working at LIGM I shared a good time with during my internship.

'What distinguishes a mathematical model from, say, a poem, a song, a portrait or any other kind of "model," is that the mathematical model is an image or picture of reality painted with logical symbols instead of with words, sounds or watercolors.' – John L. Casti

Introduction

A short history of mathematical population dynamics

In the 1750s Leonhard Euler depicts human population as a geometric series $\text{pop}_{t+1} = \lambda \cdot \text{pop}_t$ where $\lambda > 1$ represents the growth rate[1] [Bacaër, 2011]. This simply rewrites $\text{pop}_t = \lambda^t \cdot \text{pop}_0 \iff \text{pop}_t = e^{rt} \cdot \text{pop}_0$ where $r = \log(\lambda) = \mathcal{B} - \mathcal{D} > 0$ is assumed to be written as the difference between some birth and death functions, hence the idea of exponential growth. Therefore the differential equation for population at time t is:

$$\frac{\partial \text{pop}_t}{\partial t} = (\mathcal{B} - \mathcal{D}) \cdot \text{pop}_t \,.$$

In 1798, Thomas Robert Malthus pointed out the limits of this geometric growth modelization: *'the power of population is indefinitely greater than the power in the earth to produce subsistence for man'*. Inspired by earlier works and Malthus ideas, Pierre-François Verhulst proposed a new differential equation for population in 1838 [Bacaër, 2011]

$$\frac{\partial \text{pop}_t}{\partial t} = \check{r} \cdot \text{pop}_t \cdot (1 - \frac{\text{pop}_t}{m}) \,,$$

the logistic growth. Remark that this is equivalent to the previous model where $\mathcal{D}$ is replaced by $\mathcal{D}_t = \mathcal{D} \cdot \text{pop}_t$, since

$$\frac{\partial \text{pop}_t}{\partial t} = (\mathcal{B} - \mathcal{D}_t) \cdot \text{pop}_t = \mathcal{B} \cdot \text{pop}_t \cdot \left(1 - \frac{\text{pop}_t}{\mathcal{B}/\mathcal{D}}\right) \,.$$

As pop_t increases, the growth rate decreases since limiting factors appear (e.g. amount of food available), hence there is a maximal population value $m = \mathcal{B}/\mathcal{D}$ (called carrying capacity). For a small population ($\text{pop}_t \ll m$) we recover the Euler exponential growth.

Independently of one another, Lotka and Volterra investigated models taking into account several population. Alfred James Lotka suggested in 1920 that biological systems dynamics exhibit improbable and permanent oscillations [Lotka, 1920]. He considered a system in the process of evolution for a simple special case comprising two species of matter, a herbivorous animal feeding on plant. In 1925, Vito Volterra studied the predator prey fishery problem uncovered by Umberto d'Ancona in the Adriatic sea and proposed the same model as Lotka 5 years earlier.

Lotka Volterra models

Robert McCredie May, a pioneer in ecological research and theoretical analysis of population and ecosystem dynamics, by triggering an international work studying ecological communities, made it possible to progress in the study of large LV models by calling on high dimensional random matrices and miscellaneous characteristics of their spectrum. The introduction of random matrix theory into theoretical ecology in 1972 (start of the worldwide research program in theoretical ecology which is credited to R. M. May) and since then, enriches the grasp of Lotka Volterra modeling for populations dynamics. The overlaying mathematical and ecological work done on the subject enabled progress to be made on key questions about Lotka Volterra system dynamics. For a model to illustrate a viable ecosystem, the possible equilibria must be stable and feasible. During the 1970s such properties, stability and later feasibility were investigated, see e.g. [May, 1972a], [May, 1974], [Goh and Jennings, 1977]. Existence and uniqueness questions are found to be related to the Linear Complementarity Problem (LCP) introduced before 1970 in [Cottle and Dantzig, 1968]; under some conditions the LCP indeed ensures existence and unicity of a non negative equilibrium—satisfying the non invasibility condition—to the model we study, see Section 3 for its presentation.

Context and objectives

Specifically, the LCP and the Lotka Volterra system of equations examined for theoretical ecology are connected and, there exists an equivalence between the P-property of the matrix involved in the model and the unique solution of the LCP. Hence the importance of the P-matrix problem to ensure the existence of a single equilibrium to the model. This research work focuses on the effect of interaction strength within a one community ecosystem on the behavior of the model. Especially, we investigate the phase transition phenomenon of the P-property according to this interaction strength. This report is mainly a survey about the extensive reference to the theories and previous research developed on P-matrices and random matrices in Lotka Volterra models. A property transposing the P-matrix problem into a spectral characterization is highlighted and gives hope to the understanding of phase transition phenomena of the P-property in the special case of random matrices.

Organization of the report

In the very first section, we present Lotka Volterra systems for ecology and especially we focus on a well known example, the predator prey model, before introducing the role of random matrices in the field. In Section 2 we make some reminders on random matrix theory that might be useful to follow this work. We then present the model in Section 3 before highlighting in Section 4 the highest degree of research results necessary to tackle the P-matrix problem we chose to focus on. Before closing the discussion, we illustrate the research tracks taken during this project with simulations in Section 5. In this section we explain Jiri Rohn algorithm for solving the P-matrix problem, that we coded on python.

[1] growth rate = $\frac{\text{pop}_{t+1} - \text{pop}_t}{\text{pop}_t} = \lambda - 1$

1 Lotka Volterra systems for ecology

The application of Lotka Volterra models in ecology cares about understanding abundances for each species k among N species having interactions, $1 \leq k \leq N$. Then, the Lotka Volterra (LV) equation for species k describes the evolution of its abundance over time.

$$\begin{aligned}\frac{\partial x_k}{\partial t} &= x_k \left(r_k - x_k + [\Gamma_N \cdot \vec{x}]_k \right) \\ &= x_k \left(r_k - \left[(I_N - \Gamma_N) \cdot \vec{x} \right]_k \right)\end{aligned}$$

$\vec{x} = \vec{x}(t)$	abundance vector for the N species
$x_k = x_k(t)$	abundance value of species k
r_k	natural growth rate of species k
Γ_N	$N \times N$ interaction matrix

We will see in that section an example for two species in competition known as the predator prey model and then we will present a way to generalize the model to large ecosystems using random matrices.

1.1 Lotka Volterra model for lynx and snowshow hare

Figure 1: A Canadian lynx chasing a snowshoe hare. Tom & Pat Leeson – Science Photo Library

When preys are plentiful and predators have little competition, their population increases quickly. When there are too many predators, they eat all preys and the prey population declines. Then the predators starve and their population goes down again hence the prey population can recover. This model is inherently a differential equation system in which population growth depends on population stock.

We will focus on the 2×2 Lotka Volterra system of the evolution of lynx and snowshoe hare abundance as indicated by the number of pelts collected by the Hudson's Bay company historical record from 1845 to 1935 in the Mackenzie river region. Data, scanned from graph based on [Odum, 1971], come from the famous canadian lynx hare dataset [Hundley, 2003].

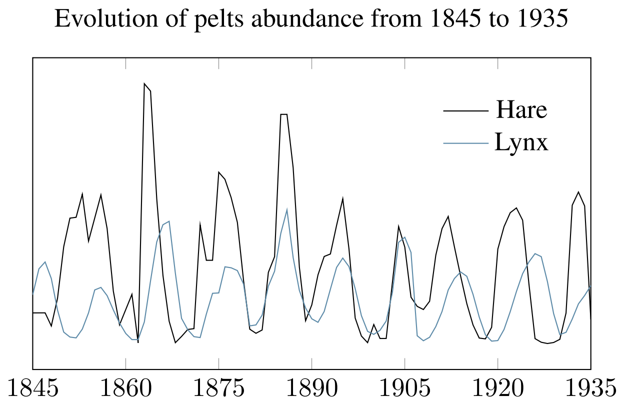


Figure 2: Illustration of the cyclical model for lynx and snowhare collected pelts based on hunting data from 1845 to 1935.

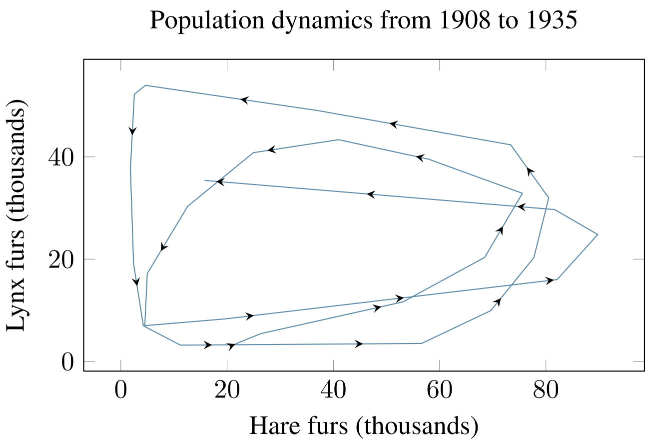


Figure 3: Illustration of the population dynamics for lynx and snowhare collected pelts based on hunting data over a time window from 1908 to 1935.

Fig. 2 and Fig. 3 show the population dynamics of Canadian lynx and snowshoe hare when the non-extinction equilibrium is unstable. Prey and predator populations tend to behave periodically when the enrichment paradox occurs, as illustrated by the oscillations in animal abundances in Fig. 2. Fig. 3 shows the plot in the (hare, lynx) space of furs headcount.

The model sets up as follows:

$$\begin{aligned}\frac{\partial h}{\partial t} &= \underbrace{a \cdot h}_{\text{increases when no predator}} - \underbrace{b \cdot h \cdot l}_{\text{transfer from hare to lynx}} \\ \frac{\partial l}{\partial t} &= - \underbrace{c \cdot l}_{\text{decreases when no catching game}} + \underbrace{d \cdot h \cdot l}_{\text{transfer from hare to lynx}}\end{aligned}$$

a, b, c, d are the parameters, h represents the hares population and l the lynxes one. One can assume a loss of mass in the transfer from hare (h) to lynx (l), leading to consider that $d \leq b$.

When looking for equilibria of the model we are investigating conditions for the process to remain unchanged, that is to say, fixed point of the differential system. Such points are called steady state, there are two ones in this model:

- 'species extinction' $\begin{cases} h &= 0 \\ l &= 0 \end{cases}$
- 'species coexist' $\begin{cases} h &= c/d \\ l &= a/b \end{cases}$

In the following we give up the cyclical model and focus on a N paired equations system, which we present in Section 3, of the form:

$$\begin{cases} \dot{x_1} = x_1 \ (r_1 - x_1 + [\Gamma_N \cdot \vec{x}]_1 \ ) \\ \dot{x_2} = x_2 \ (r_2 - x_2 + [\Gamma_N \cdot \vec{x}]_2 \ ) \\ \vdots \\ \dot{x_k} = x_k \ (r_k - x_k + [\Gamma_N \cdot \vec{x}]_k \ ) \\ \vdots \\ \dot{x_N} = x_N \ (r_N - x_N + [\Gamma_N \cdot \vec{x}]_N \ ) \end{cases}$$

1.2 Encoding reality in the interaction matrix

By launching a global research program studying ecological communities in 1972 [May, 1972b], Robert McCredie May introduced the random matrix theory into theoretical ecology. He suggested that large complex ecosystems might be modelled by random systems, whose behaviour is known. Using random matrix theory in such context enables to alleviate the difficulty to observe real interactions within large ecosystems and enriches the knowledge on Lotka Volterra models. Since then, we use normalized random matrices as the interaction matrix in the LV models.

	+	0	−
+	mutualism *bees and flowers*		
0	commensalism *small fishes hidden on sharks*	neutralism	
−	parasitism *caterpillars on oak or pine trees*	amensalism *humans and Earth*	competition *for food, shelter, partner or sunlight*

Table 1: Different kind of species interactions that can be found in nature.

Table 1 presents the different kinds of interaction that can be found in nature. For the example of parasitism, caterpillars compromise the health of oak and pine trees, while oak and pine trees have a positive effect on caterpillars by providing them with a home. The sign of the relations between two species determines the nature of their interaction.

1.2.1 i.i.d. model

The most basic assumption one can make to think about interactions between species from a diversified ecosystem, is to consider independent and identically distributed entries in the interaction matrix, $\Gamma_N = \frac{X_N}{\sqrt{N}}$ where X_N has centered i.i.d. components with unit variance (e.g. could be normally distributed with expectation 0 and variance 1, or rademacher distributed with values ± 1). Not realistic but tractable, this setting is useful for mathematical properties of random matrices. We talk about 'circular model' since eigenvalues are uniformly distributed in a disk [Girko, 1985] as we can observe on Fig. 4. Note that since the matrix from which eigenvalues are extracted is real, its complex eigenvalues will always occur in complex conjugate pairs, hence the horizontal symmetry on the graph presented in Fig. 4.

In all simulations we present in this research work, the random matrix X_N has centered components with unit variance, graphs obtained are the same for normally distributed entries with expectation 0 and variance 1, or for rademacher distributed entries with values ± 1.

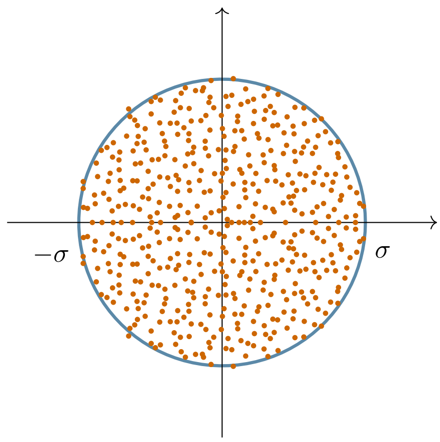


Figure 4: Uniform distribution of eigenvalues in the disk for a random matrix of size 500×500. In blue, the circle of radius equals to the entries standard deviation.

1.2.2 Elliptic model

The previous interactions modeling can be improved by linking entries from across the diagonal. The sign and magnitude of the covariance—ρ in Fig. 5 and Fig. 6—between coupled entries determines the nature, e.g. competitive or mutual (see Table 1 for more details), of the interaction.

We talk about 'elliptic model' since eigenvalues are uniformly distributed in an ellipse [Girko, 1985] as we can observe on Fig. 6. Note that, as before, complex eigenvalues occur in complex conjugate pairs, hence the horizontal symmetry on the graph presented in Fig. 6.

1.2.3 Sparse model

Within an ecosystem of N species, each species does not interact with all $N-1$ others, therefore one might want to

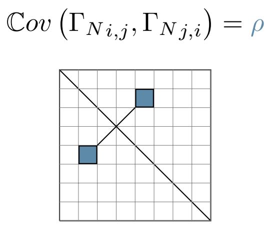


Figure 5: Illustration of the matching between symmetric entries of an elliptic matrix. Each pair of entries (meaning components with symmetric indices) comes from a multivariate distribution of size two (mean zero, unit variance, covariance equals to ρ) while the diagonal are i.i.d. with mean zero and finite variance.

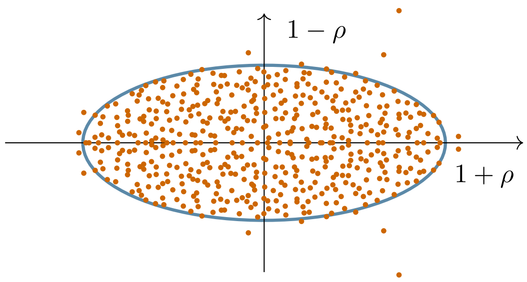


Figure 6: Uniform distribution of eigenvalues in the ellipse for an elliptic random matrix of size 500×500. In blue, the ellipse of semi major and semi minor axes equal to $1+\rho$ and $1-\rho$ respectively, ρ is the covariance of each matched components in the matrix.

consider sparse matrix of interaction, [Akjouj and Najim, 2021].

Because of the impossibility to observe the interaction matrix, we make models using large random matrices as introduced above. We can easily handle the behavior of such matrices studying the convergence of some macroscopic observables, their fluctuations or their concentration toward the mean. The asymptotics of random matrices macroscopic quantities (such as the spectral measure) allow for interesting effect on the system dynamics and thus, substantial progress in the understanding of Lotka Volterra models. In the next section, we will introduce some basics on random matrix theory before establishing our model of interest in Section 3.

2 Reminders on random matrix theory

Matrix norms. For any matrix A, we define:

- its *Frobenius norm*

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{i,j}|^2} = \sqrt{\mathrm{Tr}\left(AA^\star\right)}$$

- its *spectral norm*:

$$\|A\| = \sqrt{\lambda_{\max}\left(AA^\star\right)}$$

If A is hermitian then $\|A\| = |\lambda_{\max}(A)| \vee |\lambda_{\min}(A)|$.

Spectral radius. For a matrix B, its *spectral radius* is the maximum over its eigenvalues modulus,

$$\rho(A) = \max\left\{|\lambda|_{\mathbb{C}} \mid \lambda \in \mathrm{Sp}(A)\right\},$$

where $|\cdot|_{\mathbb{C}}$ denotes the modulus. We denote by $\rho^{\mathbb{R}}$ the spectral radius computed on real eigenvalues only.

Theorem 2.1 ([Geman, 1986, Main result]). If the random matrix A is such that $\forall i, j$,

- $\mathbb{E}[A_{i,j}] = 0$,
- $\mathbb{E}[A_{i,j}^2] = \sigma^2$,
- $\mathbb{E}[|A_{i,j}|^4] \le c$ for some positive c

then

$$\overline{\lim_{N\to\infty}} \quad \rho\left(\frac{A}{\sqrt{N}}\right) \le \sigma \quad \text{a.s.}$$

Spectral norm corresponds to the largest singular eigenvalue while spectral radius points the largest eigenvalue modulus. For random matrices with unit variance, almost surely and asymptotically spectral radius is one and largest singular value is 2 ; random matrices are an example of the gap between both spectral radius and spectral norm. Furthermore, in general we have:

Proposition 2.2. [Friedland, 2021, Lemma 2.1] $\forall A$, for any norm $\|\cdot\|$, $\rho(A) \le \|A\|$

Wigner matrix, [Bai and Yin, 1988] W_N $(N \times N)$ symmetric random matrix with i.i.d. entries such that $\mathbb{E}\left[W_{N\,i<j}\right] = 0$, $\mathbb{V}\left(W_{N\,i<j}\right) = \sigma^2$ and all entries have finite fourth moment. Variance of diagonal entries do not intervene in the results thus diagonal and off diagonal entries could have different expectation and variance (a different expectation would result in a translation of the spectrum).

- $\lambda_{\max}\left(\frac{W_N}{\sqrt{N}}\right) \xrightarrow[N\to\infty]{a.s.} 2\sigma$
- $\lambda_{\min}\left(\frac{W_N}{\sqrt{N}}\right) \xrightarrow[N\to\infty]{a.s.} -2\sigma$

Then,

$$\begin{aligned}\left\|\frac{W_N}{\sqrt{N}}\right\| &= \sqrt{\lambda_{\max}\left(\frac{1}{N} W_N W_N^\star\right)} \\ &= \left|\lambda_{\max}\left(\frac{W_N}{\sqrt{N}}\right)\right| \vee \left|\lambda_{\min}\left(\frac{W_N}{\sqrt{N}}\right)\right| \\ &\xrightarrow[N\to\infty]{a.s.} 2\sigma\end{aligned}$$

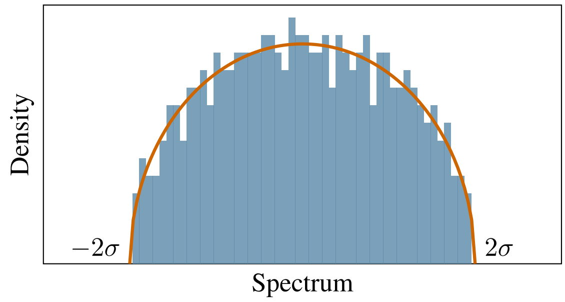


Figure 7: Eigenvalues histogram of a normalized Wigner matrix of size 500×500 fulfilled with gaussian entries (centered with variance equals to σ^2). In orange, the density of the semi circular law.

Distribution of normalized Wigner eigenvalues follows semi circular distribution $\mathcal{C}_\sigma(d\lambda) = \frac{\sqrt{(4\sigma^2-\lambda^2)_+}}{2\pi\sigma} d\lambda$ (see the shape of the eigenvalues histogram on Fig. 7).

Theorem 2.3. [Wigner, 1958] Given $\{\frac{W_N}{\sqrt{N}}\}_{N\in\mathbb{N}}$ a sequence of wigner matrices, the empirical law of eigenvalues $\mu_{W_N/\sqrt{N}} = \frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i(W_N/\sqrt{N})}$ converges asymptotically in probability to the semi circle law $\mathcal{C}_\sigma$. Especially, for all f continuous and bounded on $\mathbb{R}$,

$$\mathbb{P}\left(\left|\int f(\lambda)d\mu_{\frac{W_N}{\sqrt{N}}}(d\lambda) - \int f(\lambda)d\mathcal{C}_\sigma(d\lambda)\right|\right) \underset{N\to\infty}{\longrightarrow} 0$$

3 The model

In this section we make use of a random matrix to reflect large ecosystem interactions. We put in place a parameter in order to bring into being the interaction strength as the variance of the matrix components. While we will introduce some results on models built upon an hermitian matrix in the following sections, we will specialize on the less well studied non hermitian case.

3.1 Architecture

The present work focuses on the following LV model:

$$\frac{\partial x_k}{\partial t} = x_k\left(1 - \left[\left(I_N - \frac{X_N}{\alpha\sqrt{N}}\right)\cdot\vec{x}\right]_k\right)$$

We note $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ the interaction matrix. The interaction matrix is composed of a normalized random matrix—where entries in X_N are centered with unit variance and finite fourth moment—and a parameter α to control for the strength of interactions among the species of the ecosystem. Then the study of $\frac{X_N}{\alpha\sqrt{N}}$ comes down to the analysis of $\frac{\tilde{X}_N}{\sqrt{N}}$ where entries in $\tilde{X}_N$ are centered with variance equals to $1/\alpha^2$ and finite fourth moment.

We then define the interaction strength as $1/\alpha$; when α increases, $1/\alpha$ decreases traducing weaker interactions and vice versa.

Under this setting, the equilibrium is defined as:

$$\forall k, \quad \frac{\partial x_k{}^\star}{\partial t} = 0$$
$$\iff \forall k, \quad x_k{}^\star\left(1 - \left[\left(I_N - \frac{X_N}{\alpha\sqrt{N}}\right)\cdot\vec{x}^\star\right]_k\right) = 0$$

Then, for each k, either

- $x_k{}^\star = 0$

or

- $1 - \left[\left(I_N - \frac{X_N}{\alpha\sqrt{N}}\right)\cdot\vec{x}^\star\right]_k = 0$

3.2 Points of interest

We focus on miscellaneous properties regarding Lotka Volterra differential equation system, existence and uniqueness of an equilibrium, feasibility and stability/global stability of the latter.

3.2.1 Existence and uniqueness

We are interested in whether or not there exists an equilibrium $\vec{x}^\star$, i.e. a point from which there is no further temporal evolution:

$$\frac{\partial x_k{}^\star}{\partial t} = 0, \quad \forall k \in [\![1, N]\!] \qquad [⧗]$$

and whichever this equilibrium is unique.

3.2.2 Feasibility

Definition 3.1. The feasibility domain of an ecological community describes the set of abiotic and biotic environmental factors under which all species have positive abundances at equilibrium.

We are curious about the equilibrium feasibility, meaning whether or not there is species extinction i.e. the equilibrium is feasible if

$$x_k{}^\star > 0, \quad \forall k \in [\![1, N]\!] \qquad [⊗]$$

If $I_N - \Gamma_N$ is invertible the feasible equilibrium (when all $x_k{}^\star$ are different from zero) is defined as

$$\vec{x}^\star = (I_N - \Gamma_N)^{-1} \cdot \mathbb{1}_N$$

Theorem 3.2. If a matrix A is such that $\rho(A) < 1$ then $I - A$ is invertible and $(I - A)^{-1} = \sum_{k\geq 0} A^k$

Proof . *First, note that* $\rho(A) = \lim_{K\to\infty} \left\|A^K\right\|^{1/K}$ *[Gelfand, 1941]. By this formula,* $\rho(A) < 1 \implies \lim_{K\to\infty} \left\|A^K\right\| < 1 \implies \lim_{K\to\infty} A^K = 0.$

$$\begin{aligned}
\sum_{k=0}^{K} A^k &= I + A + \cdots + A^K \\
A\sum_{k=0}^{K} A^k &= A + A^2 + \cdots + A^{K+1} \\
\sum_{k=0}^{K} A^k - A\sum_{k=0}^{K} A^k &= I - A^{K+1} \\
\iff (I - A)\sum_{k=0}^{K} A^k &= I - A^{K+1} \\
\lim_{K\to\infty} (I - A)\sum_{k=0}^{K} A^k &= I - \underbrace{\lim_{K\to\infty} A^{K+1}}_{0 \;\; since \;\; \rho(A)<1} \\
(I - A)\sum_{k\geq 0} A^k &= I
\end{aligned}$$

Therefore $(I - A)$ is invertible and its inverse is defined as $(I - A)^{-1} = \sum_{k\geq 0} A^k$.

Thus, $I_N - \Gamma_N$ is invertible if $\rho(\Gamma_N) < 1$. By Theorem 2.1 $\rho(\Gamma_N) \leq \frac{1}{\alpha}$ almost surely for N large, which is strictly inferior to one if and only if $\alpha > 1$. Then, if $\alpha > 1$, $I_N - \Gamma_N$ is invertible and both criteria [⧗] and [⊗] are equivalent to $\vec{x}^\star = (I_N - \Gamma_N)^{-1} \cdot \mathbb{1}_N$.

[Bizeul and Najim, 2021] examine a Lotka Volterra model involving an interaction strength term decreasing with the dimension of the ecosystem ($\alpha_N \underset{N\to\infty}{\longrightarrow} \infty$, $\alpha_N = \kappa\sqrt{\log N}$, $\kappa > 0$). Feasibility of the equilibrium is investigated and a phase transition for the componentwise positivity of the equilibrium vector of abundances is highlighted at $\alpha_N{}^\star = \sqrt{2\log N}$.

Theorem 3.3 (Bizeul & Najim)**.**

- If $\exists\, \varepsilon > 0$ such that $\alpha_N \leq (1-\varepsilon)\alpha_N{}^\star$ then $\mathbb{P}\left(\min_{k\in[\![1,N]\!]} x_k{}^\star > 0\right) \underset{N\to\infty}{\longrightarrow} 0$
- If $\exists\, \varepsilon > 0$ such that $\alpha_N \geq (1+\varepsilon)\alpha_N{}^\star$ then $\mathbb{P}\left(\min_{k\in[\![1,N]\!]} x_k{}^\star > 0\right) \underset{N\to\infty}{\longrightarrow} 1$

The heuristic to understand the critical scaling lies on the fact that the expected value of the minimum over N i.i.d. gaussian variables is $-\sqrt{2\log N}$.

Using our definition of the interaction matrix—see the LV model—and assuming that $I_N - \Gamma_N$ is invertible (which is true if $\rho(\Gamma_N) < 1$, i.e. $\alpha > 1$), then:

$$\begin{aligned}
\vec{x}^\star &= (I_N - \Gamma_N)^{-1} \cdot \mathbb{1}_N \\
\forall k \in [\![1, N]\!],\; x_k{}^\star &= \left[(I_N - \Gamma_N)^{-1} \cdot \mathbb{1}_N\right]_k \\
&= \left[\sum_{l\geq 0} (\Gamma_N)^l \cdot \mathbb{1}_N\right]_k \\
&\approx 1 + \frac{1}{\alpha_N} \underbrace{\sum_{i\in[\![1,N]\!]} \frac{[X_N]_{k,i}}{\sqrt{N}}}_{Z_k\sim\mathcal{N}(0,1)} + \ldots \\
\text{hence} \quad \min_{k\in[\![1,N]\!]} x_k{}^\star &\approx 1 + \frac{1}{\alpha_N} \min_{k\in[\![1,N]\!]} Z_k + \ldots \\
&\approx 1 - \frac{\sqrt{2\log N}}{\alpha_N}
\end{aligned}$$

The crux of the demonstration for the critical scaling $\alpha_N{}^\star = \sqrt{2\log N}$ at which the feasibility property becomes true, relies on the negligence of the remaining terms (designated by $\ldots$ in the above).

3.2.3 Stability and global stability

Definition 3.4. An ecosystem is stable if it returns to its equilibrium state after a disturbance in the abundance of its species due to various environmental fluctuations.

We are concerned about the equilibrium stability i.e. whether the steady state attracts surrounding points toward itself or repels neighborhood points.

An equilibrium point is (locally) stable if, by placing the system somewhere near the point, the latter will evolve towards the equilibrium point. Global stability means that whatever the initial point of the system, it will reach the equilibrium point.

For non negative equilibrium to be stable it is required that,

$$1 - [(I_N - \Gamma_N) \cdot \vec{x}^{\star}]_k \leq 0,\ \forall k \in [\![1, N]\!] \qquad [✋]$$

$$\iff \frac{1}{x_k}\frac{\partial x_k}{\partial t}\bigg|_{x_k \to 0^+} \leq 0,\ \forall k \in [\![1, N]\!]$$

(see [Law and Morton, 1996, "*Condition for invasion by a new specie*"]). Such equilibrium is said to be *saturated*.

In ecology this requirement is better known as the non invasibility condition and we include it as an hypothesis in our LV model. It traduces the fact that when a small population of a species is introduced into the environment it does not proliferate or, when the size of a population drops it cannot bounce back.

In his book "*Global dynamical properties of Lotka Volterra systems*", Yasuhiro Takeuchi studies similar systems to the LV model and devotes a whole chapter to their stability properties. At this stage we need to introduce a new concept to handle Takeuchi's results.

Definition 3.5. A is said to be *Volterra Lyapunov stable* if there exists a positive definite diagonal matrix D such that $DA + A^{\star}D$ is symmetric negative definite.

The main result of this chapter on global stability of the LV system equilibrium is the following:

Theorem 3.6 (Takeuchi & Adachi, [Takeuchi, 1996, Theorem 3.2.1])**.** The Lotka Volterra system

$$\dot{x}_k = x_k\Big(r_k - \sum_{l=1}^{N} A_{k,l}x_l\Big)$$

for $k \in [\![1, N]\!]$ has a non negative and globally stable equilibrium point $x^{\star}$ for each $r \in \mathbb{R}^N$ if $-A$ is Volterra Lyapunov stable.

4 Literature review on the state of the art

We evoked previously the main theoretical results on feasibility and global stability of the equilibrium. In this section, we will first see how these properties and the existence and uniqueness criteria are related to the well known Linear Complementarity Problem, then we will focus on the P-property and see how we can translate the algebraic definition of a P-matrix into a spectral characterization.

4.1 Lotka Volterra systems and the Linear Complementarity Problem

Finding an equilibrium point to the Lotka Volterra system is equivalent to solve a specific Linear Complementarity Problem (LCP). The LCP is a well known system of inequalities which has a wide range of applications from optimization theory to mathematical programming. Its objective is to find out solutions $x, y \in \mathbb{R}^N$ such that

$$\begin{aligned} &y = Ax + r \geq 0,\\ &x \geq 0,\\ &y^T \cdot x = 0 \end{aligned}$$

for A a $N \times N$ matrix and $r \in \mathbb{R}^N$. We note this system LCP(A, r).

In the previous section, we have seen conditions for $\vec{x}^{\star}$ to be an equilibrium point [⧗] which does not result in any species extinction: feasibility criterion [⊗] and we have seen the non invasibility assumption [✋].

Then, seeking a non negative equilibrium point to our LV system is equivalent to solve the LCP$(I_N - \Gamma_N, -\mathbb{1}_N)$ finding the abundances vector $\vec{x}^{\star}$ satisfying:

$$\begin{aligned} &[✋] \quad (I_N - \Gamma_N) \cdot \vec{x}^{\star} - \mathbb{1}_N \geq 0,\\ &[⊗] \quad \vec{x}^{\star} \geq 0,\\ &[⧗] \quad ((I_N - \Gamma_N)\,\vec{x}^{\star} - \mathbb{1}_N)^T \cdot \vec{x}^{\star} = 0 \end{aligned}$$

4.1.1 LCP solution

The connection between our problem and the LCP$(I_N - \Gamma_N, -\mathbb{1}_N)$ motivates the importance puts on the existence of solution(s) to the LCP and the uniqueness, the feasibility of the possible solution and the global stability of the latter. As follows, the LCP$(I_N - \Gamma_N, -\mathbb{1}_N)$ is closely connected to the P-property of the matrix $I_N - \Gamma_N$.

> **Definition 4.1.** *Principal minors* of a matrix are the determinants of the principal submatrices obtained when striking out a same set of rows and columns.

See below (Fig. 8) an example of principal submatrix construction.

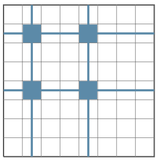


Figure 8: Representation of principal submatrix components (in blue) from a matrix obtained when selecting the same indices in row and column.

> **Definition 4.2.** A is said to be a *P-matrix* if all its principal minors are strictly positive.[2]

> **Theorem 4.3** ([Murty, 1972, Theorem 4.2]). LCP(A, r) has a unique solution for each $r \in \mathbb{R}^N$ if and only if A is a P-matrix.

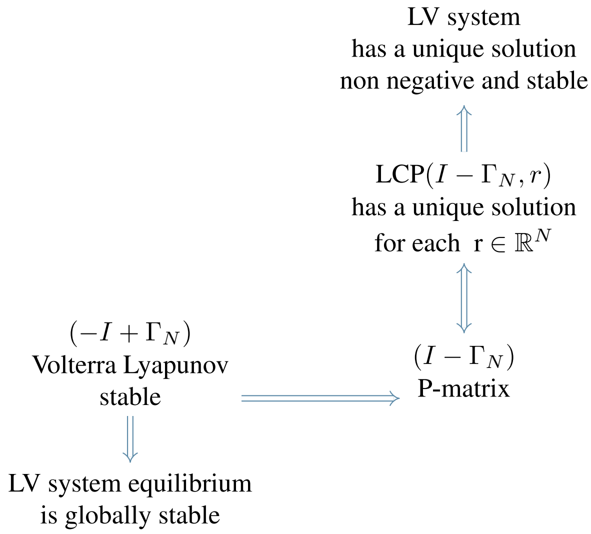
Thus the P-property of the matrix $I_N - \Gamma_N$ is equivalent to existence and uniqueness of a non negative equilibrium, fulfilling the non invasibility requirement, to the Lotka Volterra system for all natural growth rate vector $r \in \mathbb{R}^N$. Especially it would imply that LCP$(I_N - \Gamma_N, -\mathbb{1}_N)$ has a unique solution, in other words it ensures existence and uniqueness of a non negative equilibrium to the Lotka Volterra system for the natural growth rate equals to $\mathbb{1}_N$.

Furthermore, the property of being a P-matrix is itself a consequence of the Volterra Lyapunov stability introduced earlier.

> **Theorem 4.4** (Takeuchi, Adachi and Tokumaru; [Takeuchi et al., 1978, Theorem 2] and [Takeuchi, 1996, Lemma 3.2.1]). If $-A$ is Volterra Lyapunov stable, then A is a P-matrix and the real parts of its eigenvalues are positive.

Moreover we know from [Cottle et al., 2009, Theorem 3.3.4] that a P-matrix has positive real eigenvalues. Thus, if $-A$ is Volterra Lyapunov stable, its spectrum is located on the right part of the complex plane.

From Theorems 4.3 and 4.4 we get the following summary:



In Section 4.2, we will investigate the conditions on interaction strength to satisfy the P-property of the random matrix of interactions. First, we study how previous results are reflected in conditions on the interaction strength for the feasibility and the global stability of the possible solution to the LCP.

4.1.2 Feasibility condition

There is a phase transition phenomenon on the interaction strength for the feasibility of the equilibrium in the Lotka Volterra system (for both, hermitian or non hermitian, random matrix of interaction), [Bizeul and Najim, 2021, Theorem 1.1] at $\frac{1}{\alpha} = \frac{1}{\sqrt{2 \log N}}$.

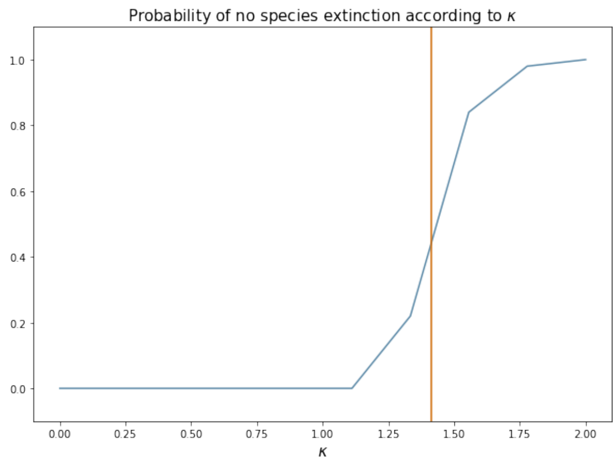


Figure 9: Simulation of the probability of no species extinction. Phase transition at $\kappa = \sqrt{2}$. For each value $\kappa \in (0, 2]$ the value of the curve corresponds to a Montecarlo simulation over 50 iterations for $\Gamma_N = \frac{X_N}{\kappa\sqrt{N \log N}}$ of size 500×500. We get the same graph for both hermitian and non hermitian matrices.

On Fig. 9 we can observe the phase transition phenomenon proved in [Bizeul and Najim, 2021, Theorem

[2] Note that each diagonal entry of a P-matrix which represent a submatrix determinant when deleting all other rows and columns are among the principal minors. The determinant of the matrix, which is the determinant of a submatrix equal to the matrix itself, is also among the principal minors.

1.1].

4.1.3 Global stability condition

The assumption of non invasibility presented earlier is necessary to ensure stability of the possible non negative equilibrium of the model but not sufficient. However, global stability is fulfilled if the matrix $\Gamma_N - I_N$ is Volterra Lyapunov stable. The phase transition for the global stability is not proved. However we know from [Takeuchi, 1996, Chapter 3] a sufficient condition for the convergence of the LV model toward a non negative globally stable equilibrium: the Volterra Lyapunov stability. In other words, if there exists a matrix D diagonal positive definite such that $D(-I+\Gamma_N) + (-I+\Gamma_N)^\star D < 0$—in the symmetric negative definite sense—then global stability of the non negative equilibrium is ensured.

Non hermitian case. Considering the normalized random matrix $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ where X_N has centered entries with unit variance, we check this condition for the positive definite diagonal matrix $D = I$.

$$\begin{aligned}&(-I+\Gamma_N) + (-I+\Gamma_N)^\star < 0\\ \iff &-2I + \Gamma_N + \Gamma_N^\star < 0\\ \iff &\lambda_{\max}(\Gamma_N + \Gamma_N^\star) < 2\end{aligned}$$

Moreover,

- $[\Gamma_N + \Gamma_N^\star]_{i,j} = \frac{X_{N\,i,j} + X_{N\,j,i}}{\alpha\sqrt{N}} = [\Gamma_N + \Gamma_N^\star]_{j,i}$
- $\mathbb{V}([\Gamma_N + \Gamma_N^\star]_{i,j}) = \begin{cases}\frac{2}{\alpha^2 N} & \text{for } i \neq j\\ \frac{4}{\alpha^2 N} & \text{for } i = j\end{cases}$

Therefore $\Gamma_N + \Gamma_N^\star = \frac{1}{\sqrt{N}}\begin{bmatrix}\ddots & \\ & \ddots\end{bmatrix}$ is a normalized Wigner matrix and $\lambda_{\max}(\Gamma_N + \Gamma_N^\star) \xrightarrow[N\to\infty]{a.s.} \frac{2\sqrt{2}}{\alpha}$. Hence

$$\begin{aligned}&\frac{1}{\alpha} < \frac{1}{\sqrt{2}}\\ \iff &\lambda_{\max}(\Gamma_N + \Gamma_N^\star) < 2 \quad \text{a.s. for } N \text{ large}\\ \iff &I_N - \Gamma_N \text{ Volterra Lyapunov stable a.s. for } N \text{ large}\end{aligned}$$

Then for α below $\sqrt{2}$ the Volterra Lyapunov stability cannot be shown using I, however this does not ensure that $I_N - \Gamma_N$ is not Volterra Lyapunov stable a.s. eventually. We do not want to investigate the Volterra Lyapunov stability with $D \neq I$ since we cannot apply classical random matrix theory result because of the far more complex variance profile. In particular, [Bunin, 2017, Discussion] corroborates the idea of a transition phase phenomenon on the interaction strength for the global stability of the equilibrium around $\frac{1}{\alpha} = \frac{1}{\sqrt{2}}$ (for a non hermitian matrix).

Hermitian case. Considering the normalized symmetric random matrix $\Gamma_N = \frac{W_N}{\alpha\sqrt{N}}$ being a Wigner matrix with centered entries having variance equals to $\frac{1}{\alpha^2}$, we note that $\lambda_{\max}\left(\frac{W_N}{\alpha\sqrt{N}}\right) \xrightarrow[N\to\infty]{a.s.} \frac{2}{\alpha}$. From [Berman and Hershkowitz, 1983, Theorem 1]:

> **Theorem 4.5.** For symmetric matrices, the Volterra Lyapunov stability is equivalent to the P-property.

From [Cottle et al., 2009, Proposition 2.2.16]:

> **Proposition 4.6.** A symmetric matrix is a a P-matrix if and only if it is positive definite.

We are looking for the necessary and sufficient conditions on interaction strength $\frac{1}{\alpha}$ for having $I - \Gamma_N$ a positive definite matrix when N is large enough. Since this is an hermitian matrix, its eigenvalues are real. Thus, we are looking for conditions on the eigenvalues to be asymptotically positive. Hence

$$\begin{aligned}&\frac{1}{\alpha} < \frac{1}{2}\\ \iff &\lambda_{\max}\left(\frac{W_N}{\alpha\sqrt{N}}\right) < 1 \quad \text{a.s. for } N \text{ large}\\ \iff &\forall k \quad 0 < \lambda_k(I - \Gamma_N) \quad \text{a.s. for } N \text{ large}\end{aligned}$$

Therefore, for an hermitian matrix, there is a transition phase phenomenon on the interaction strength for the global stability of the equilibrium around $\frac{1}{\alpha} = \frac{1}{2}$.

4.2 The P-property

A consequence of Theorem 4.3 is that if $I_N - \Gamma_N$ is a P-matrix, then the LV system has a unique non negative equilibrium. Hence the importance of identifying conditions on interactions strength to satisfy the P-property.

Thanks to Theorem 4.5 and Proposition 4.6 there is a phase transition phenomenon for both, the P-property and the Volterra Lyapunov stability, at $\frac{1}{\alpha} = \frac{1}{2}$ in the context of an hermitian random matrix. However, [Rohn and Rex, 1996, Theorem 3.4] show that the problem is NP-hard for a general real matrix.

4.2.1 Towards a spectral characterization

We have no result for non hermitian random matrices that could participate to solve the P-matrix problem. Nevertheless as we will see in this part, the P-matrix problem is related to the regularity of an interval matrix, [Rump, 2003a, Theorem 2.1].

> **Definition 4.7.** An *interval matrix* $[\underline{A}, \overline{A}]$ is defined as
>
> $$[\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\},$$
>
> where the inequality is componentwise. $[\underline{A}, \overline{A}]$ is said to be *regular* if it contains no singular matrix; otherwise it is *singular*.

We shall often consider the center matrix and the radius, $A_c = \frac{\overline{A}+\underline{A}}{2}$ and $\Delta = \frac{\overline{A}-\underline{A}}{2}$ so that, $[\underline{A}, \overline{A}] = [A_c \pm \Delta]$.

Theorem 4.8. If $A - I$ and $A + I$ are non singular then the following properties are equivalent:

(i) A is a P-matrix;
(ii) $[(A-I)^{-1}(A+I) \pm I]$ is regular;
(iii) $\max_{x \in \{\pm 1\}^n} \rho^{\mathbb{R}} \left((A+I)^{-1} (A-I) \operatorname{diag}(x) \right) < 1$.

Proof . *Define* $A_c = (A-I)^{-1}(A+I) \iff A = (A_c+I)(A_c-I)^{-1}$*. This theorem stems from [Rump, 2003a, Theorem 2.1] applied with the good center matrix (the* A *in the expression of [Rump, 2003a, Theorem 2.1] corresponds to* ${A_c}^{-1}$ *here) and a sign real spectral radius bound* r *equals to one. Then, to see that both first points, for the theorem and the consequence, are the same, we need to observe that*

$$\begin{aligned} &\left(I - {A_c}^{-1}\right)^{-1} \left(I + {A_c}^{-1}\right) \\ &= \left(I - {A_c}^{-1}\right)^{-1} + \left(I - {A_c}^{-1}\right)^{-1} {A_c}^{-1} \\ &= (A_c + I)(A_c - I)^{-1} \\ &= A \end{aligned}$$

Indeed

$$\begin{aligned} \left(I - {A_c}^{-1}\right)^{-1} &= \left(A_c {A_c}^{-1} - I {A_c}^{-1}\right)^{-1} \\ &= \left((A_c - I) {A_c}^{-1}\right)^{-1} \\ &= A_c (A_c - I)^{-1} \end{aligned}$$

hence

$$\begin{aligned} \left(I - {A_c}^{-1}\right)^{-1} {A_c}^{-1} &= \left(A_c \left(I - {A_c}^{-1}\right)\right)^{-1} \\ &= \left(A_c I - A_c {A_c}^{-1}\right)^{-1} \\ &= (A_c - I)^{-1} \end{aligned}$$

Therefore, the P-property of $I_N - \Gamma_N$ (provided that the matrix Γ_N is non singular) is equivalent to the regularity of the interval matrix

$$\left[-2{\Gamma_N}^{-1} + I_N \pm I_N\right] = \left[-2{\Gamma_N}^{-1}, -2{\Gamma_N}^{-1} + 2I_N\right].$$

Note that in our case, matrices belonging to the interval are positive diagonal perturbations of the matrix $-2{\Gamma_N}^{-1}$.

Considering any interval matrix of the form $[A_c \pm \Delta]$, we introduce a new term called the regularity radius, which represents the "distance" from the center matrix to singularity, [Poljak and Rohn, 1993, Theorem 2.1], [Rohn, 2012b, definition 3.3.24].

Definition 4.9. We define the *regularity radius* as

$$d(A_c, \Delta) = \min\{\delta \geq 0; [A_c \pm \delta\Delta] \text{ is singular}\}.$$

[Poljak and Rohn, 1993, Theorem 2.1] introduce a simpler version of the regularity radius:

$$d(A_c, \Delta) = \frac{1}{\max_{y,z \in \{\pm 1\}^n} \rho^{\mathbb{R}} \left[{A_c}^{-1} \operatorname{diag}(y) \Delta \operatorname{diag}(z)\right]}$$

In this definition $\operatorname{diag}(y), \operatorname{diag}(z)$ are diagonal matrices filled with $y, z \in \{\pm 1\}^n$. Then for our $\Delta = I_N$, $\operatorname{diag}(y)\Delta \operatorname{diag}(z) = \operatorname{diag}(x)$ for $x \in \{\pm 1\}^n$.

$$d(A_c, I) = \frac{1}{\max_{x \in \{\pm 1\}^n} \rho^{\mathbb{R}} \left[{A_c}^{-1} \operatorname{diag}(x)\right]}$$

Therefore, the P-property of $I_N - \Gamma_N$ is equivalent to $d\left(-2{\Gamma_N}^{-1} + I_N, I_N\right) > 1$ which corresponds to the third point in Theorem 4.8. Such condition illustrates that our interval matrix does not touch the singularity border, which ensures regularity of our interval. Note that $\max_{x \in \{\pm 1\}^n} \rho^{\mathbb{R}} (\cdot \operatorname{diag}(x))$ is often called the "sign real spectral radius", see e.g. [Rump, 2003a], [Rohn, 2012c]. In [Rohn, 2012c, theorem 2], the author proves that the first two points of the equivalence in Theorem 4.8 can be proved under the unique assumption that $A - I$ is non singular. [Rohn and Shary, 2018, Theorem 9] proves a purely linear algebraic result about regularity radius. Considering an interval matrix centered on A_c

- either $d\left({A_c}^{-1}, I\right) \leq 1$
- or $d(A_c, I) \leq 1$

In the same paper [Rohn and Shary, 2018, Theorem 5], authors state that if $[I \pm |A_c|]$ or $[{A_c}^{-1} \pm I]$ is regular, then $[A_c \pm I]$ is singular. Hence [Rohn and Shary, 2018, Theorem 9]:

- either $[{A_c}^{-1} \pm I]$ is singular that is, $\exists S$ singular $\in [{A_c}^{-1} \pm I] \iff d\left({A_c}^{-1}, I\right) \leq 1$
- or it is regular which implies using [Rohn and Shary, 2018, Theorem 5] that $[A_c \pm I]$ is singular that is, $\exists S$ singular $\in [A_c \pm I] \iff d(A_c, I) \leq 1$

Another formulation is

- either $d\left({A_c}^{-1}, I\right) \leq 1$
- or $d\left({A_c}^{-1}, I\right) > 1 \implies d(A_c, I) \leq 1$

Unfortunately we do not have the other implication:

$$\begin{aligned} & d(A_c, I) \leq 1 \implies d\left({A_c}^{-1}, I\right) > 1 \\ \iff & d\left({A_c}^{-1}, I\right) \leq 1 \implies d(A_c, I) > 1 \end{aligned}$$

5 Simulations

Based on simulations and theoretical results found in the literature the hope of this research work was to put evidence on a phase transition phenomenon for the P-property at $\frac{1}{\alpha} = 1$ in the context of a non hermitian random matrix. Indeed, we would have liked to show:

$\forall \varepsilon > 0,\ \exists N^\star$ such that $\forall N \geq N^\star$, $(1+\varepsilon) I - \Gamma_N$ is a P-matrix a.s. $\iff \alpha > 1$.

Miscellaneous paths of ideas have been investigated to show the above result,

- working on the spectral radius of submatrices to apply [Geman, 1986, Main result],
- working on interval matrices to show that the spectrum of all matrices of matter (namely, positive diagonal perturbations of $-2{\Gamma_N}^{-1}$ contained in the interval of interest) do not include 0,
- finding conditions so that the sign real spectral radius, and the regularity radius in turn, would ensure the P-property of our matrix of interest

5.1 Spectral radius and determinant of submatrices

Given a realization ω of a sequence of random matrices $\{X_N\}_{N\in\mathbb{N}}$, whose entries follow a symmetric distribution (centered with variance equals to σ^2), we obtain from [Geman, 1986] that there is a $\omega \in \tilde{\Omega} \subset \Omega$ so that $\mathbb{P}(\tilde{\Omega}) = 1$ and for all $\varepsilon > 0$, there exists a $N^\star(\omega, \varepsilon)$ such that for all $N \geq N^\star$,

$$\rho\left(\frac{X_N}{\sqrt{N}}\right) \leq \sigma + \varepsilon\,.$$

Then we have been able to show (Appendix A.1) that for a fixed set of indices $\mathcal{I} \subset [\![1, N]\!]$, for all $\varepsilon > 0$, there exists $N^\star(\omega, \varepsilon, \mathcal{I})$ such that for all $N \geq N^\star$ the result is true. While $N^\star$ depends on the submatrix indices $\mathcal{I}$, simulations (Fig. 10) support the idea that there might exist $N^\star$ so that for all N above and for each $\mathcal{I} \subset [\![1, N]\!]$ the spectral radius of the submatrix is bounded by σ.

On Fig. 10 note that for all entries distribution (real gaussian, complex gaussian, rademacher ± 1) we see the same pattern, increasing maximal spectral radius and decreasing minimal determinant according to the dimension. Remark that there is no horizontal symmetry between the maximal spectral radius of a submatrix and the minimal determinant of the latter. Indeed,

$$\min_{\mathcal{I}\subset[\![1,N]\!]} \det\left(I - \frac{{X_N}^{\mathcal{I}}}{\sqrt{N}}\right) = \min_{\mathcal{I}\subset[\![1,N]\!]} \prod_{k\in\mathcal{I}} \left(1 - \lambda_k\left(\frac{{X_N}^{\mathcal{I}}}{\sqrt{N}}\right)\right)$$
$$\neq \prod_{k\in\mathcal{I}} \left(1 - \max_{\mathcal{I}\subset[\![1,N]\!]} \rho\left(\frac{{X_N}^{\mathcal{I}}}{\sqrt{N}}\right)\right)$$

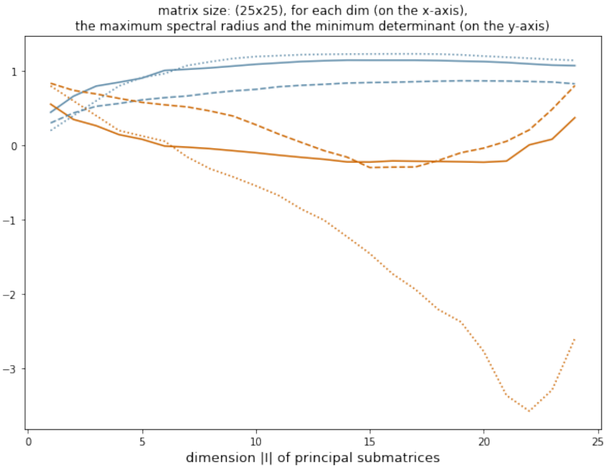


Figure 10: Evolution of the maximal spectral radius (in blue) and the minimal determinant (in orange) among all principal submatrices dimension, for 3 kind of 25×25 random matrices:
solid line: real Gaussian entries with $\mu = 0, \sigma = 1$,
dashed line: complex Gaussian entries with $\mu = 0, \sigma = 1$,
dotted line: Rademacher $\{\pm 1\}$ entries all normalized by $\sqrt{N}$.

Also, some determinants are negative but it does not rebut our idea since in the simulation $\alpha = 1$, the matrix is small, there is no reason for it to be a P-matrix.

Then, motivated by the simulations (Fig. 10) that exhibit an increasing behavior of principal submatrices spectral radius according to their dimension, what we would have liked to prove is the following (see Conjecture 5.1):

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measured space. There exists $\tilde{\Omega} \subset \Omega$ such that $\mathbb{P}(\tilde{\Omega}) = 1$ and for a realization $\omega = \{\Gamma_N\}_{N\in\mathbb{N}} \in \tilde{\Omega}$ which is a sequence of $(N \times N)$ random matrices, for all $\varepsilon > 0$ there exists a $N^\star(\omega, \varepsilon)$ such that for all $N \geq N^\star$ and $\mathcal{I} \subset [\![1, N]\!]$, $\det([(\sigma+\varepsilon)I - \Gamma_N]^{\mathcal{I}}) > 0$.

5.2 Spectrum evolution under a positive diagonal perturbation

Random matrices are isotropic. Therefore, instead of studying $-2{\Gamma_N}^{-1} + \Delta$, Δ being a positive diagonal matrix with entries in $[0, 2]$, we will consider ${\Gamma_N}^{-1} + \Delta$, where Δ has diagonal entries in $[0, 1]$.

On Fig. 11 we observe the well known spectrum of a random matrix contained in the disk (left) and, the one of its inverse outside of the disk (right).

We consider four kinds of diagonal deformation (where diagonal entries of Δ might be superior to one in order to show the deformation pattern occurring when the deformation grows):

- ${\Gamma_N}^{-1} + \Delta$ where Δ is $\delta \times I$ with $\delta \in [0, 3]$,
- a sparse version of the same deformation (with $1/3$ of the diagonal entries set to zero) with $\delta \in [0, 5]$,
- ${\Gamma_N}^{-1} + \Delta$ where Δ has distinct entries in $[0, 3]$,
- a sparse version of the same deformation (with $1/3$ of the diagonal entries set to zero) with $\delta \in [0, 5]$

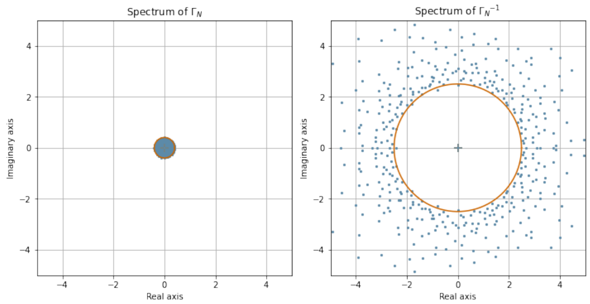


Figure 11: Spectrum of the 500×500 random matrix $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ and ${\Gamma_N}^{-1}$ where $\alpha = \frac{1}{0.4}$. The radius for Γ_N is the standard deviation that is $\frac{1}{\alpha} = 0.4$ and the one for ${\Gamma_N}^{-1}$ is its inverse.

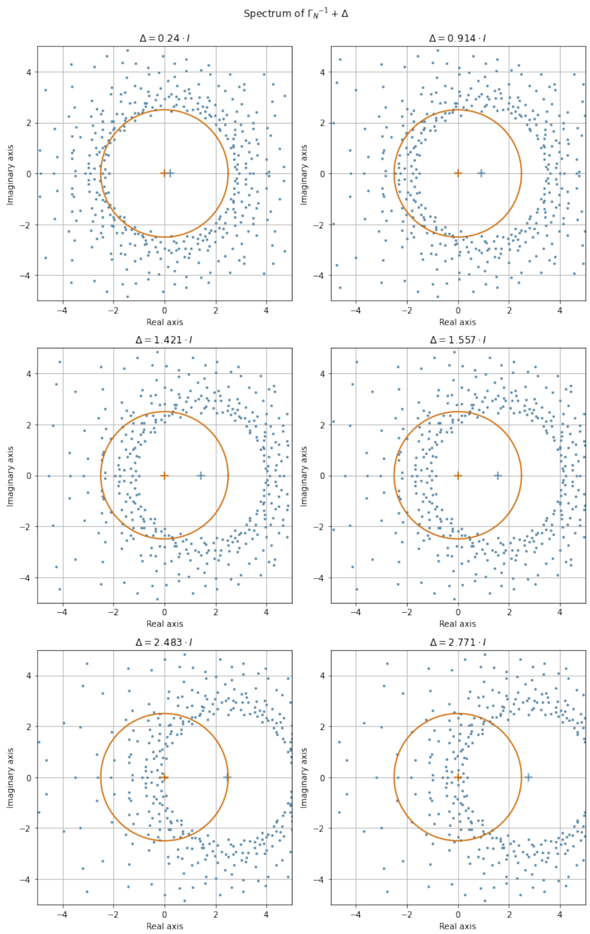


Figure 12: Evolution of the spectrum for a 500×500 random matrix inverse ${\Gamma_N}^{-1}$ perturbed by $\delta \cdot I$ with $\delta \in [0, 3]$. $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ where $\alpha = \frac{1}{0.4}$. In orange, the circle of radius equals to the inverse of the standard deviation, that is to say α in this case. In blue, the eigenvalues of ${\Gamma_N}^{-1} + \Delta$.

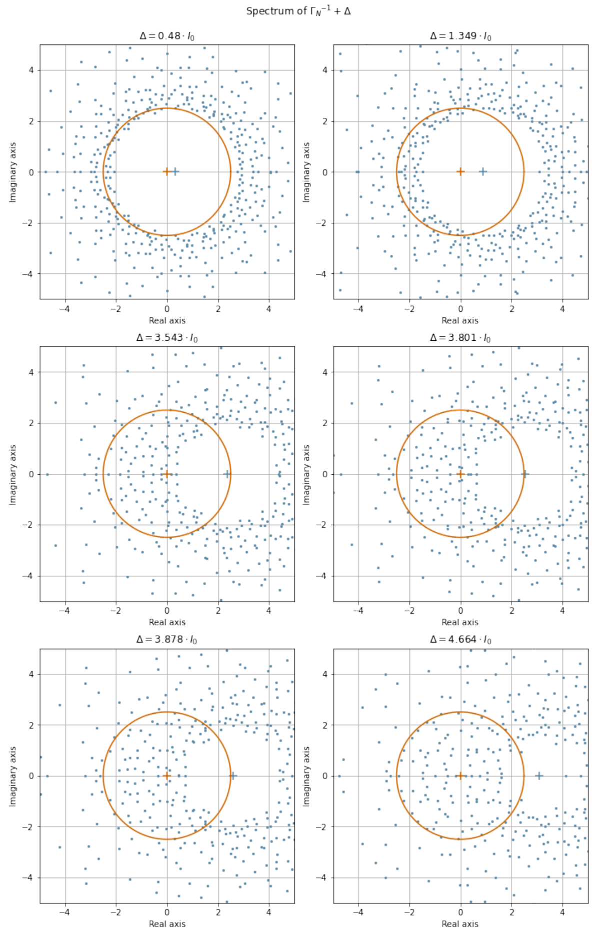


Figure 13: Evolution of the spectrum for a 500×500 random matrix inverse ${\Gamma_N}^{-1}$ perturbed by a sparse version of $\delta \cdot I$ with $\delta \in [0, 5]$ (a third of zeros). $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ where $\alpha = \frac{1}{0.4}$. In orange, the circle of radius equals to the inverse of the standard deviation, that is to say α in this case. In blue, the eigenvalues of ${\Gamma_N}^{-1} + \Delta$.

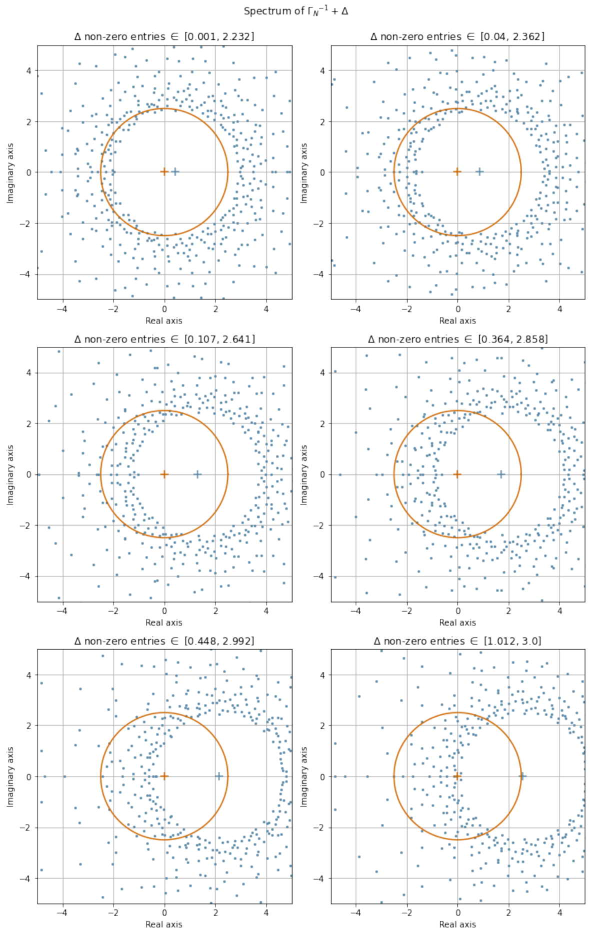


Figure 14: Evolution of the spectrum for a 500×500 random matrix inverse ${\Gamma_N}^{-1}$ perturbed by a diagonal with distinct entries in $[0, 3]$. $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ where $\alpha = \frac{1}{0.4}$. In orange, the circle of radius equals to inverse of the standard deviation, that is to say α in this case. In blue, the eigenvalues of ${\Gamma_N}^{-1} + \Delta$.

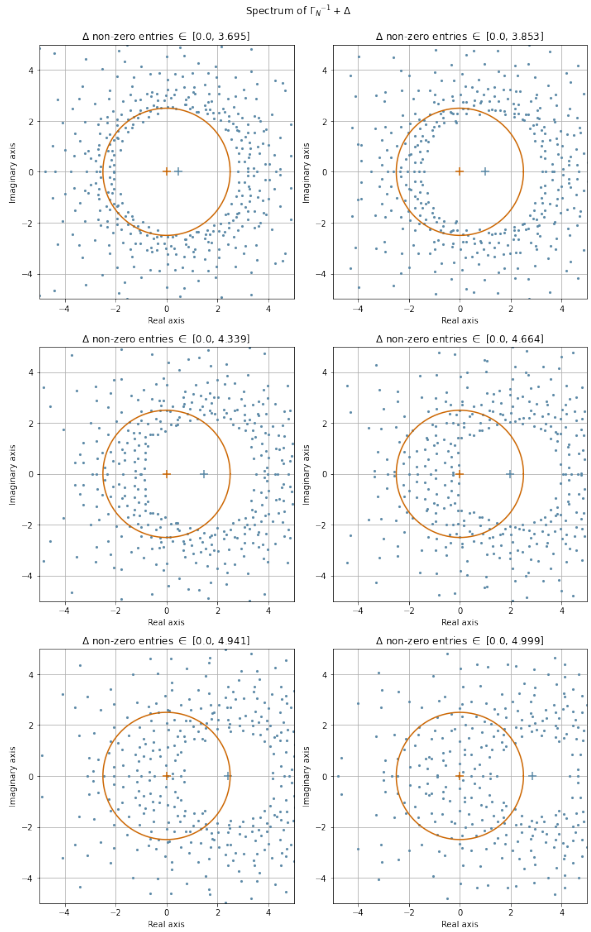


Figure 15: Evolution of the spectrum for a 500×500 random matrix inverse ${\Gamma_N}^{-1}$ perturbed by a sparse diagonal with distinct entries in $[0, 5]$ (a third of zeros). $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ where $\alpha = \frac{1}{0.4}$. In orange, the circle of radius equals to inverse of the standard deviation, that is to say α in this case. In blue, the eigenvalues of ${\Gamma_N}^{-1} + \Delta$.

In Figs. 12 to 15, the perturbation matrix Δ involved has its entries increasing among each subgraph.

We observe two main phenomena when considering positive diagonal perturbations Δ of the random matrix inverse ${\Gamma_N}^{-1}$, namely concentration and translation of the spectrum.

While the deformation proportional to the identity and the deformation built on distinct diagonal entries translate the spectrum positively (to the right), see Fig. 12 and Fig. 14 respectively, sparse versions of these deformations exhibit a concentration of the spectrum in addition to its translation, see Fig. 13 and Fig. 15.

5.3 Exploration on the sign real spectral radius

Fig. 16 presents the evolution of the maximum real spectral radius of $(-2{\Gamma_N}^{-1} + I_N)^{-1} \operatorname{diag}(x)$ (over 50 iterations, for matrix size $= 500 \times 500$) for specific $\operatorname{diag}(x)$ with diagonal entries in ± 1. It is interesting to see that the maximal values appears when $\operatorname{diag}(x)$ is reversing the sign of almost no rows of $(-2{\Gamma_N}^{-1} + I_N)^{-1}$ or almost all

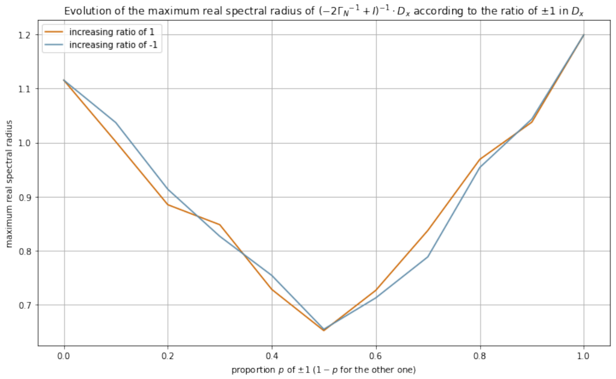


Figure 16: Evolution of $\max \rho^{\mathbb{R}}[(-2{\Gamma_N}^{-1} + I_N)^{-1} \operatorname{diag}(x)]$ for different $\operatorname{diag}(x)$ diagonal matrices of different ratio of ± 1, for a 500×500 generated random matrix. In red (resp. in black) is represented the evolution of the maximum real spectral radius computed on 50 iterations according to the ratios p (resp. $1-p$) of $+1$ (resp. -1) on the diagonal $\operatorname{diag}(x)$.

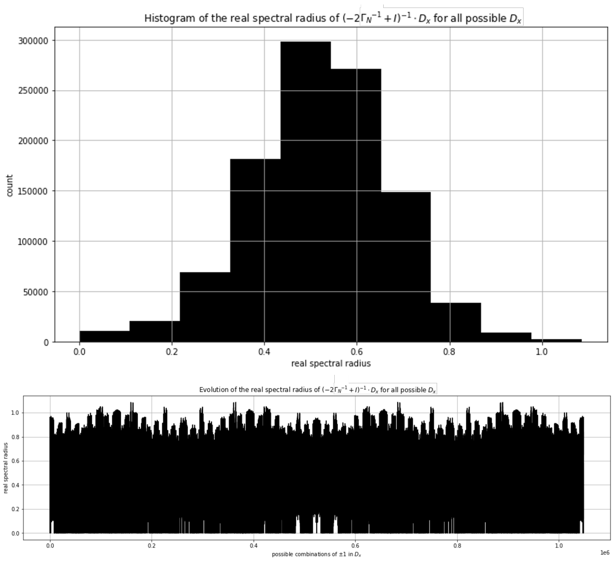


Figure 17: Histogram (top) and evolution (bottom) of the values taken by $\rho^{\mathbb{R}}[(-2{\Gamma_N}^{-1} + I_N)^{-1} \operatorname{diag}(x)]$ for all possible $\operatorname{diag}(x)$ diagonal matrices filled with ± 1, for a 20×20 generated random matrix. This graph supports the results of the above one with the idea that, in our context, $\lim_{n\to\infty} \sup \rho^{\mathbb{R}}[(-2{\Gamma_N}^{-1} + I_N)^{-1} \operatorname{diag}(x)] \leq 1$.

rows of $(-2{\Gamma_N}^{-1} + I_N)^{-1}$. When some rows have their sign reversed the real spectral radius seems to be almost surely smaller.

However, when we look at all the possible $\operatorname{diag}(x)$ on Fig. 17 (for smaller matrix size: 20×20 since it is really computationally expansive) we cannot infer that $\max \rho^{\mathbb{R}}[(-2{\Gamma_N}^{-1} + I_N)^{-1} \operatorname{diag}(x)]$ will occur for $\operatorname{diag}(x)$ filled with large proportion of either $+1$ or -1 (see the plot of real spectral radius possible values); this is confirmed by a little investigation on the proportion of ± 1 for large values of the real spectral value. Nonetheless we can see that a few values of the spectral radius are either small (0) or large (1), while a lot are around 0.5 (see the histogram).

The miscellaneous paths of ideas we have been through did not succeed to alleviate the difficulty encountered to prove the phase transition phenomenon of the P-matrix. However, the possible transition at $\frac{1}{\alpha} = 1$ is corroborated by simulations done on small matrices (because of the expensive computations needed to solve the P-matrix problem) thanks to Rohn's algorithm as we will see in the next subsection. Further, in Section 5.5 we will investigate elements for evidence of the phase transition phenomenon.

5.4 Rohn Algorithm

In the technical report [Rohn, 2012a], the author proposes a matlab pseudo code to detect the P-property. Matlab code can be found in [Rohn, 2016]. A translation of this work into python can be found at `https://github.com/robachowyk/RMT-LVeq-Ecology/tree/main/check_p_property`.

Algorithm 1 Solving the P-matrix problem

Require: A
  $n = \texttt{shape}(A)$
  $I = \texttt{eye}(n)$
Ensure: $\texttt{rank}(A - I) = n$
  $A_c = (A - I)^{-1}(A + I)$
  $S = \texttt{regising}(A_c, I)$
  if S is empty **then return** $True$
  else return $False$

This algorithm is based on [Rohn, 2012c, Theorem 2]. The $\texttt{regising}(A_c, \Delta)$ program considers an exhaustive list of methods to determine the REGularIty or the SINGularity of the interval $[A_c - \Delta, A_c + \Delta]$. [Rohn, 2012c, Theorem 2] states that for $(A - I)$ non-singular, A is a P-Matrix if and only if the interval

$$[(A - I)^{-1}(A + I) - I, (A - I)^{-1}(A + I) + I]$$

is regular. This interval matrix is of the form $[A_c - \Delta, A_c + \Delta]$ with $A_c = (A - I)^{-1}(A + I)$ and $\Delta = I$, hence we call $\texttt{regising}(A_c, I)$ in Algorithm 1. `regising` checks the regularity/singularity of the interval matrix by returning a matrix S singular, if one has been found in the interval matrix (i.e. singular interval) or a value $S = [\,]$ empty if no singular matrix has been found in the interval (i.e. regular interval). It investigates the following methods:

Conditions for the existence of a singular matrix.

- *midpoint matrix A_c:* the midpoint matrix A_c of the interval matrix $[A_c \pm \Delta]$ is singular.
- *diagonal condition* [Rex and Rohn, 1998, Theorem 2.1]: $|A_c x| \leq \Delta |x|$ has a non trivial (i.e. non zero) solution x.
- *steepest determinant descent* [Rohn, 1989, Algorithm 5.1]: investigate determinant bounds of the interval matrix (i.e. the hull of matrices determinant for matrices in the interval).

- *two Qz-matrices* [Jansson and Rohn, 1999, Theorem 4.3]: the linear programming problem

$$\begin{aligned} \text{maximize} \quad & z^T x \\ \text{subject to} \quad & (A_c - \Delta\,\text{diag}(z))x \leq 0 \\ & \text{diag}(z)x \geq 0 \end{aligned} \qquad [\star]$$

  is unbounded for some $z \in \{\pm 1\}^n$.
- *main algorithm* [Rohn, 1993, Theorem 2.2]: loop on $\{\pm 1\}^n$ to identify the possible singular matrix which should have the specific form.
- *symmetrization* [Rex and Rohn, 1998, Sections 4 and 5]: both of the following conditions imply the singularity of $[A_c \pm \Delta]$:
  1. $\lambda_{\max}({A_c}^T A_c) \leq \lambda_{\min}(\Delta^T \Delta)$
  2. $\Delta^T \Delta - {A_c}^T A_c$ positive definite

Conditions for the regularity of the interval.

- *Beeck's condition* [Rex and Rohn, 1998, Corollary 3.2 from Beeck]: $\rho(|{A_c}^{-1}|\Delta) < 1$ is regular (for A_c non singular).
- *symmetrization* [Rex and Rohn, 1998, Sections 4 and 5]: both of the following conditions imply the regularity of $[A_c \pm \Delta]$:
  1. $\lambda_{\max}(\Delta^T \Delta) < \lambda_{\min}({A_c}^T A_c)$
  2. ${A_c}^T A_c - \|\Delta^T \Delta\| I$ is positive definite
- *two Qz-matrices* [Jansson and Rohn, 1999, Theorem 4.3]: the linear programming problem Eq. ($\star$) is bounded for all $z \in \{\pm 1\}^n$.
- *main algorithm* [Rohn, 1993, Theorem 2.2]: loop on $\{\pm 1\}^n$ to check there is no singular matrix in the whole interval. This last track is the most expensive since in case the matrix is a P-matrix and no one of the conditions presented above succeeded to prove it, the algorithm will investigate the values of the sign real spectral radius.

5.5 Phase transition phenomenon

Because of the last NP-hard subcase of the algorithm presented, it is very expansive to apply the P-matrix algorithm on large random matrices. Therefore we investigated the phase transition for the P-property on the parameter α for 15×15 matrices only. Below are the graphs obtained, for a non hermitian matrix (for which the phase transition is conjectured to occur at $\alpha = 1$ but has not yet been showed), Fig. 18. For an hermitian matrix the phase transition has been proved to occur at $\alpha = 2$.

These simulations support the following conjecture, expressed in a more general way:

Conjecture 5.1. Let $\frac{X_N}{\alpha\sqrt{N}}$ be a normalized random matrix, centered with unit variance and bounded fourth moments. For all $\varepsilon > 0$ we consider $\left(\frac{1}{\beta} + \varepsilon\right) I_N - \frac{X_N}{\alpha\sqrt{N}}$, and we conjecture that

- if $\frac{1}{\beta} < \frac{1}{\alpha}$, then $\left(\frac{1}{\beta} + \varepsilon\right) I_N - \frac{X_N}{\alpha\sqrt{N}}$ is not a P-matrix
- if $\frac{1}{\beta} > \frac{1}{\alpha}$, then $\left(\frac{1}{\beta} + \varepsilon\right) I_N - \frac{X_N}{\alpha\sqrt{N}}$ is a P-matrix

Being a P-matrix implies to have positive real eigenvalues, [Cottle et al., 2009, Theorem 3.3.4]. Therefore, $\forall \lambda_k \in \mathbb{R}$, $\frac{1}{\beta} + \varepsilon - \frac{\lambda_k\left(\frac{X_N}{\sqrt{N}}\right)}{\alpha}$ should be positive. Since such real eigenvalue: $\lambda_k\left(\frac{X_N}{\sqrt{N}}\right)$ belongs to $[-1, 1]$ (due to the convergence of eigenvalues distribution towards a circular law) a.s. for N large, and are uniformly distributed on this interval Edelman et al. [1994]

Erratum: Edelman only proves a convergence in distribution which is too weak for us.

$$\frac{1}{\beta} + \varepsilon - \frac{\lambda_k\left(\frac{X_N}{\sqrt{N}}\right)}{\alpha} > 0 \text{ for each } \lambda_k\left(\frac{X_N}{\sqrt{N}}\right) \text{ real}$$
$$\iff \frac{1}{\beta} + \varepsilon > \frac{1}{\alpha}$$

Erratum: This proof lacks one important argument, we are not sure that a.s. for N large the maximal eigenvalue would converge towards 1.

Thus $\frac{1}{\beta} + \varepsilon < \frac{1}{\alpha} \implies \left(\frac{1}{\beta} + \varepsilon\right) I_N - \frac{X_N}{\alpha\sqrt{N}}$ is not a P-matrix . However, proving that

$$\frac{1}{\beta} + \varepsilon > \frac{1}{\alpha} \implies \left(\frac{1}{\beta} + \varepsilon\right) I_N - \frac{X_N}{\alpha\sqrt{N}} \text{ is a P-matrix}$$

is another kettle of fish.

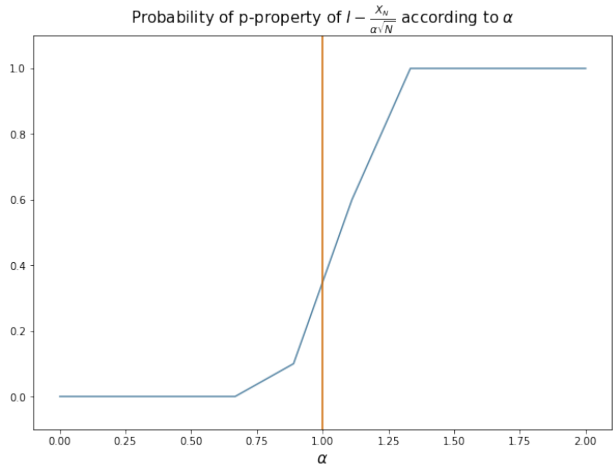


Figure 18: Simulation of the probability of being a P-Matrix based on Rohn's algorithm, [Rohn, 2016, Regularity/Singularity of an interval matrix algorithm], [Rohn, 2012a]. Phase transition at $\alpha = 1$. For each value $\alpha \in (0, 2]$ the value of the curve corresponds to a Monte-carlo simulation over 10 iterations for non-hermitian random matrix (i.i.d. reduced centered gaussian) $\Gamma_N = \frac{X_N}{\alpha\sqrt{N}}$ of size 15×15.

Conclusion

While the stability and feasibility questions in the context of large Lotka Volterra models involving random matrices have been addressed in the literature, the P-property is less well investigated. This property is indeed complex to check, or even impossible on a researcher timescale in case of large interaction matrix, which may have discouraged some. At least two main algorithms exist to probe the P-property of a matrix, [Tsatsomeros and Li, 2000], [Rohn, 2012a]. However the P-matrix problem still remains NP-hard, which often makes computations too expansive to run these algorithms.

The spectral characterization of the P-property studied in [Rump, 2003a], [Rohn, 2012c] and presented in the last part of Section 4 provides hope to field in the context of random matrix theory. Indeed, random matrices exhibit an interesting spectral behavior and conditions might be found within such background in order to enhance algorithms and prove or disprove the supposed phase transition according to the interaction strength within the ecological community studied by the Lotka Volterra model.

A Appendix

A.1 Ideas based on Geman spectral radius bound

As seen in Section 5, Geman's result can be generalized to the principal submatrices.

> **Proposition A.1.** X_N a random matrix (whose inputs follow a symmetric distribution) such that $\forall i, j$, $\mathbb{E}\left[X_{N\,i,j}\right] = 0, \mathbb{V}\left(X_{N\,i,j}\right) = \sigma^2, \mathbb{E}\left[\left|X_{N\,i,j}\right|^4\right] \leq c$ for some c. Then [Geman, 1986] bound on the spectral radius is also true for some principal submatrix: $\mathcal{I} \subset [\![1, N]\!]$, $\overline{\lim\limits_{N\to\infty}} \quad \rho\left(\frac{X_N{}^{\mathcal{I}}}{\sqrt{N}}\right) \leq \sigma$ a.s.

Proof . *The spectral radius of a random matrix X_N normalized by $\sqrt{N}$, whose entries follow a symmetric distribution, has a well known $\lim\limits_{N\to\infty}$ sup almost surely since [Geman, 1986, Main result]. Given a realization ω of a sequence of such random matrices $\{X_N\}_{N\in\mathbb{N}}$, we thus have $\omega \in \tilde{\Omega} \subset \Omega$ so that $\mathbb{P}\left(\tilde{\Omega}\right) = 1$, $\forall\varepsilon > 0$, $\exists N^\star(\omega,\varepsilon)$ such that $\forall N \geq N^\star$, $\rho\left(\frac{X_N}{\sqrt{N}}\right) \leq \sigma + \varepsilon$. This last sentence rewrites $\forall N \geq N^\star$, $\lim\limits_{p\to\infty} \left\|\frac{X_N^p}{N^{p/2}}\right\|^{1/p} \leq \sigma + \varepsilon$, [Gelfand, 1941], where $\|\cdot\| = \sqrt{\sum_{i,j} |\cdot|^2_{i,j}} = \sqrt{\mathrm{Tr}\left(\cdot^\star\,\cdot\right)}$ denotes the Frobenius norm for matrices.*

We want to transpose this result to $X_N^{\mathcal{I}}$ a principal submatrix of X_N for some $\mathcal{I} \subset [\![1, N]\!]$ fixed. Based on Geman's proof we will show that $\forall\varepsilon > 0$, $\exists N^\star(\omega,\varepsilon,\mathcal{I})$ such that $\forall N \geq N^\star$ -which now also depends on the sequence of submatrix indices-the result is true. For this purpose, for any $\rho > \sigma$ and for all positive integer $p \leq 1$[3], we will develop $\mathbb{E}\left[\sum_{N\geq 1} \frac{\|(\cdot)^p\|^2}{\rho^{2p}}\right]$ for the matrix and the principal submatrix.

$$\mathbb{E}\left[\sum_{N\geq 1} \frac{\left\|\left(\frac{X_N{}^{\mathcal{I}}}{\sqrt{N}}\right)^p\right\|^2}{\rho^{2p}}\right]$$

$$= \sum_{N\geq 1} \frac{1}{\rho^{2p}N^p}\mathbb{E}\left[\left\|\left(X_N{}^{\mathcal{I}}\right)^p\right\|^2\right]$$

$$= \sum_{N\geq 1} \frac{1}{\rho^{2p}N^p}\mathbb{E}\left[\sum_{i,j}[(X_N{}^{\mathcal{I}})^p]^2_{i,j}\right]$$

$$= \sum_{N\geq 1} \frac{1}{\rho^{2p}N^p}\mathbb{E}\left[\sum_{i,j}[(X_N{}^{\mathcal{I}})^p]_{i,j}[(X_N{}^{\mathcal{I}})^p]_{i,j}\right] \text{ [4]}$$

$$= \sum_{N\geq 1} \frac{1}{\rho^{2p}N^p} \underbrace{\sum_{\substack{i,j,\\ k_1,\dots,k_{p-1},\\ l_1,\dots,l_{p-1}}}}_{2p \text{ indices } \in \mathcal{I}} \mathbb{E}\,[X_N{}^{\mathcal{I}}]_{i,k_1}[X_N{}^{\mathcal{I}}]_{k_1,k_2}$$

$$\dots [X_N{}^{\mathcal{I}}]_{k_{p-2},k_{p-1}}[X_N{}^{\mathcal{I}}]_{k_{p-1}j}[X_N{}^{\mathcal{I}}]_{i,l_1}[X_N{}^{\mathcal{I}}]_{l_1,l_2}$$

$$\dots [X_N{}^{\mathcal{I}}]_{l_{p-2},l_{p-1}}[X_N{}^{\mathcal{I}}]_{l_{p-1}j}$$

$$= \sum_{N\geq 1} \frac{1}{\rho^{2p}N^p} \quad (*)$$

$$\mathbb{E}\left[\sum_{N\geq 1} \frac{\left\|\left(\frac{X_N}{\sqrt{N}}\right)^p\right\|^2}{\rho^{2p}}\right]$$

$$= \sum_{N\geq 1} \frac{1}{\rho^{2p}N^p} \underbrace{\sum_{\substack{i,j,\\ k_1,\dots,k_{p-1},\\ l_1,\dots,l_{p-1}}}}_{2p \text{ indices } \in [\![1,N]\!]} \mathbb{E}\,[X_N]_{i,k_1}[X_N]_{k_1,k_2}$$

$$\dots [X_N]_{k_{p-2},k_{p-1}}[X_N]_{k_{p-1}j}[X_N]_{i,l_1}[X_N]_{l_1,l_2}$$

$$\dots [X_N]_{l_{p-2},l_{p-1}}[X_N]_{l_{p-1}j}$$

$$= \sum_{N\geq 1} \frac{1}{\rho^{2p}N^p} \quad (**)$$

When expressing a principal submatrix of Γ_N, we still consider the normalization by $\sqrt{N}$ while the principal submatrices have smaller size than $(N \times N)$. This important point allows us to express $(**)$ according to $(*)$ since $\frac{X_N{}^{\mathcal{I}}}{\sqrt{N}} \subset \frac{X_N}{\sqrt{N}}$.

$$\mathbb{E}\left[\sum_{N\geq 1}\frac{\left\|\left(\frac{X_N}{\sqrt{N}}\right)^p\right\|^2}{\rho^{2p}}\right]$$

$$= \sum_{N\geq 1}\frac{1}{\rho^{2p}N^p} \quad (**)$$

$$= \sum_{N\geq 1}\frac{1}{\rho^{2p}N^p} \quad (*)$$

$$+ \sum_{N\geq 1}\frac{1}{\rho^{2p}N^p}\underbrace{\sum_{i,j,\dots}}_{\text{indices } \in [\![1,N]\!]\setminus\mathcal{I}}\mathbb{E}[X_N]_{i,\cdot}\dots[X_N]_{\cdot,j}$$

$$[X_N]_{i,\cdot\cdot}\dots[X_N]_{\cdot\cdot,j}$$

In the second term above, $\underbrace{\sum_{i,j,\dots}}_{\text{indices } \in [\![1,N]\!]\setminus\mathcal{I}}\mathbb{E}[X_N]_{i,\cdot}\dots[X_N]_{\cdot,j}[X_N]_{i,\cdot\cdot}\dots[X_N]_{\cdot\cdot,j}$ *focuses on the expectation of terms from the whole matrix* X_N *that are not considered in the principal submatrix* ${X_N}^{\mathcal{I}}$*. For each element in the sum, if any term appear an odd number of times, it makes the element of the sum vanish as odd moments of symmetric law variables are zero. Therefore, the second term in the above expression only sums products of even moments. As a consequence this is a positive term, therefore:*

$$\mathbb{E}\sum_{N\geq 1}\frac{\left\|\left(\frac{{X_N}^{\mathcal{I}}}{\sqrt{N}}\right)^p\right\|^2}{\rho^{2p}} \leq \mathbb{E}\sum_{N\geq 1}\frac{\left\|\left(\frac{X_N}{\sqrt{N}}\right)^p\right\|^2}{\rho^{2p}}$$

By [Geman, 1986] proof the right term is finite which guarantees $\sum_{N\geq 1}\frac{\|({X_N}^{\mathcal{I}}/\sqrt{N})^p\|^2}{\rho^{2p}} < \infty$ *a.s. Then by Hadamard criterion for power series* $\overline{\lim_{p\to\infty}}\left\|\left({X_N}^{\mathcal{I}}/\sqrt{N}\right)^p\right\|^2 \leq \rho^{2p}$ *a.s. hence our result:* $\overline{\lim_{p\to\infty}}\left\|\left(\frac{{X_N}^{\mathcal{I}}}{\sqrt{N}}\right)^p\right\|^{1/p} \leq \rho$ *a.s. Because this is true for any fixed* $\rho > \sigma$ *we get that* $\overline{\lim_{N\to\infty}}\rho\left(\frac{{X_N}^{\mathcal{I}}}{\sqrt{N}}\right) \leq \sigma$ *a.s.*

A.2 Additional results

For positive square real matrix we know from [Rump, 2003b, Theorem 2.9]:

[2] $\mathbb{C}^+ = \{z \in \mathbb{C};\ Im(z) > 0\}$

[3] In [Geman, 1986] proof, $\forall N,\ p = p_N \geq 1, p_N \sim k\log N$ and $\sum_{N\geq 1} N^2 p_N/\rho^{2p_N} < \infty$. We can therefore chose the same ρ and the same p when developing the expectations formulas for $\Gamma_N^{\mathcal{I}}$ and for Γ_N.

Theorem A.2. For A with non negative entries, $\rho(A) < \sigma \iff \sigma I - A$ is a P-matrix

To prove it, we apply [Horn and Johnson, 1994, Theorem 2.5.3] on the Z-matrix $\sigma I - A$[5]. If $\rho(A) \leq \sigma$ in such representation, A is said to be a M-matrix [Jeyaraman and Sivakumar, 2016, Definitions page 43]. Then, if this inequality is strict : $\rho(A) < \sigma$, A is a nonsingular M-matrix, that is to say A is a P-matrix [Plemmons, 1977, Theorem 1].

From [Rump, 1997, Theorem 2.3], we have the following equivalences for A a $(n \times n)$ matrix and b positive:

Theorem A.3. $\max_{x\in\{\pm 1\}^n}\rho^{\mathbb{R}}(A\,\mathrm{diag}(x)) < b \iff \forall x \in \{\pm 1\}^n\ bI - \mathrm{diag}(x)A$ is a P-matrix

From [Lehmann et al., 2017, Theorem 2.1] we have for A a real matrix:

Theorem A.4. $\max_{x\in\{\pm 1\}^n}\rho^{\mathbb{R}}(A\,\mathrm{diag}(x)) < 1 \iff z \mapsto z + A|z|$ is bijective in $\mathbb{R}^n$

From [Barker et al., 1978, Theorem of Lyapunov], for a matrix A we have:

Theorem A.5. The eigenvalues of A have positive real parts $\iff \exists H$ symmetric positive definite such that $AH + HA^\star$ is positive definite.

A.3 Simulation tests

We have seen in Section 4 that regarding the interval matrix $[A_c \pm I]$

- either $d\left({A_c}^{-1}, I\right) \leq 1$
- or $d\left({A_c}^{-1}, I\right) > 1 \implies d\left(A_c, I\right) \leq 1$

But unfortunately the converse is not proved. The other implication we are interested in would have been the following:

$$d\left(A_c, I\right) \leq 1 \implies d\left({A_c}^{-1}, I\right) > 1$$
$$\text{i.e. } d\left({A_c}^{-1}, I\right) \leq 1 \implies d\left(A_c, I\right) > 1$$

Which traduces in:

$$\frac{1}{\max_{x\in\{\pm 1\}^n}\rho^{\mathbb{R}}\left[A_c\,\mathrm{diag}(x)\right]} \leq 1 \implies \frac{1}{\max_{x\in\{\pm 1\}^n}\rho^{\mathbb{R}}\left[{A_c}^{-1}\,\mathrm{diag}(x)\right]} > 1$$

Then,

$$\max_{x\in\{\pm 1\}^n}\rho^{\mathbb{R}}\left[A_c\,\mathrm{diag}(x)\right] \geq 1 \implies \max_{x\in\{\pm 1\}^n}\rho^{\mathbb{R}}\left[{A_c}^{-1}\,\mathrm{diag}(x)\right] < 1$$

Continuing this reasoning, algorithmically speaking if $\rho^{\mathbb{R}}\left[A_c\right] \geq 1$ then $d\left(A_c, I\right) > 1$; thus we would get an additional condition to check that $[A_c \pm I]$ is regular.

[5] Z-matrices have the form $\sigma I - P$ with $P \geq 0$ and $\sigma > 0$ [Fiedler and Markham, 1992, Definition 1.1]

J. Rohn algorithm for testing the P-property does not work for large matrices ($n > 15$). If we find conditions for which this implication is true, we could re arrange the algorithm by including the conditions so that our matrix of interest Γ_N micht be checked to be a P-matrix.

It is quite interesting to see what happen if we insert this condition in the algorithm (without evidence of relevance). We recover the phase transition at $\alpha \approx \sqrt{2}$ proposed by the physicists for global stability [Bunin, 2017] (see Fig. 19).

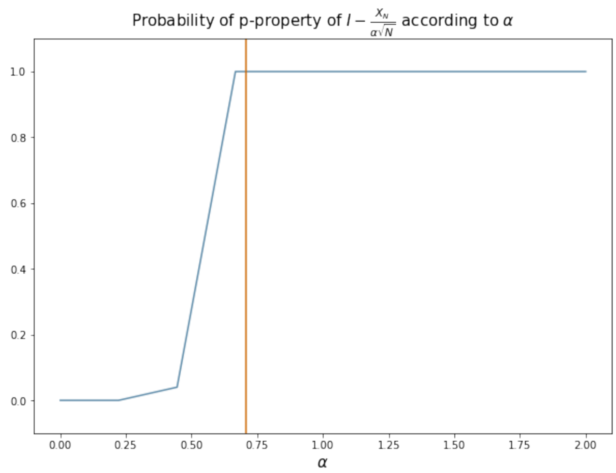


Figure 19: Simulation of the probability of being a P-Matrix based on Rohn's algorithm (re arranged). Phase transition at $\alpha \approx \frac{1}{\sqrt{2}}$. For each value $\alpha \in (0, 2]$ the value of the curve corresponds to a Montecarlo simulation over 50 iterations for non hermitian random matrix (i.i.d. reduced centered gaussian) of size 100×100.

With the correct algorithm we find a transition phase at $\alpha = 1$ (but we cannot test the algorithm for large matrices) - see Fig. 18.

References

Akjouj, I. and Najim, J. (2021). Feasibility of sparse large lotka-volterra ecosystems.

Bacaër, N. (2011). *Lotka, Volterra and the predator–prey system (1920–1926)*, pages 71–76. Springer London, London.

Bai, Z. D. and Yin, Y. Q. (1988). Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a wigner matrix. *The Annals of Probability*, 16(4):1729–1741.

Barker, G. P., Berman, A., and Plemmons, R. J. (1978). Positive diagonal solutions to the lyapunov equations. *Linear and Multilinear Algebra*, 5(4):249–256.

Berman, A. and Hershkowitz, D. (1983). Matrix diagonal stability and its implications. *Siam Journal on Algebraic and Discrete Methods*, 4.

Bizeul, P. and Najim, J. (2021). Positive solutions for large random linear systems. *Proceedings of the American Mathematical Society*, page 1.

Bunin, G. (2017). Ecological communities with lotka-volterra dynamics. *Phys. Rev. E*, 95:042414.

Cottle, R. W. and Dantzig, G. B. (1968). Complementary pivot theory of mathematical programming. *Linear Algebra and its Applications*, 1(1):103–125.

Cottle, R. W., Pang, J.-S., and Stone, R. E. (2009). *The Linear Complementarity Problem*. Society for Industrial and Applied Mathematics.

Edelman, A., Kostlan, E., and Shub, M. (1994). How many eigenvalues of a random matrix are real? *Journal of the American Mathematical Society*, 7(1):247–267.

Fiedler, M. and Markham, T. L. (1992). A classification of matrices of class z. *Linear Algebra and its Applications*, 173:115–124.

Friedland, S. (2021). Infimum of a matrix norm of a induced by an absolute vector norm. *arXiv*.

Gelfand, I. (1941). Normierte ringe. *Matematicheskii Sbornik*.

Geman, S. (1986). The Spectral Radius of Large Random Matrices. *The Annals of Probability*, 14(4):1318 – 1328.

Girko, V. L. (1985). Circular law. *Theory of Probability & Its Applications*, 29(4):694–706.

Goh, B. and Jennings, L. (1977). Feasibility and stability in randomly assembled lotka-volterra models. *Ecological Modelling*, 3(1):63–71.

Horn, R. A. and Johnson, C. R. (1994). *Topics in Matrix Analysis*. Cambridge University Press, Cambridge; New York.

Hundley, D. R. (2003). Introduction to mathematical modelling. `http://people.whitman.edu/~hundledr/courses/M250F03/M250.html`. The Famous Canadian Lynx-Hare Data.

Jansson, C. and Rohn, J. (1999). An algorithm for checking regularity of interval matrices. *SIAM Journal on Matrix Analysis and Applications*, 20(3):756–776.

Jeyaraman, I. and Sivakumar, K. (2016). Complementarity properties of singular m-matrices. *Linear Algebra and its Applications*, 510:42–63.

Law, R. and Morton, R. D. (1996). Permanence and the assembly of ecological communities. *Ecology*, 77(3):762–775.

Lehmann, L., Radons, M., Rump, S. M., and Strohm, C. (2017). Sign controlled solvers for the absolute value equation with an application to support vector machines.

Lotka, A. J. (1920). Analytical note on certain rhythmic relations in organic systems. *Proceedings of the National Academy of Sciences*, 6(7):410–415.

May, R. M. (1972a). Will a large complex system be stable? *Nature*.

May, R. M. (1972b). Will a large complex system be stable? *Nature*, 238.

May, R. M. (1974). *Stability and Complexity in Model Ecosystems*, volume 1. Princeton University Press.

Murty, K. G. (1972). On the number of solutions to the complementarity problem and spanning properties of complementary cones. *Linear Algebra and its Applications*, 5(1):65–108.

Odum, E. P. (1971). *Fundamentals of Ecology*. Saunders Comp.

Plemmons, R. (1977). M-matrix characterizations.i—nonsingular m-matrices. *Linear Algebra and its Applications*, 18(2):175–188.

Poljak, S. and Rohn, J. (1993). Checking robust nonsingularity is np-hard. *Mathematics of Control, Signals and Systems*, 6.

Rex, G. and Rohn, J. (1998). Sufficient conditions for regularity and singularity of interval matrices. *SIAM Journal on Matrix Analysis and Applications*, 20(2):437–445.

Rohn, J. (1989). Systems of linear interval equations. *Linear Algebra and its Applications*, 126:39–78.

Rohn, J. (1993). Interval matrices: Singularity and real eigenvalues. *Siam Journal on Matrix Analysis and Applications - SIAM J MATRIX ANAL APPLICAT*, 14.

Rohn, J. (2012a). An algorithm for solving the p-matrix problem. Technical report.

Rohn, J. (2012b). A manual of results on interval linear problems. Technical report.

Rohn, J. (2012c). On rump's characterization of p-matrices. *Optimization Letters*, 6.

Rohn, J. (2016). Regularity/singularity of an interval matrix. Technical report.

Rohn, J. and Rex, G. (1996). Interval p-matrices. *SIAM Journal on Matrix Analysis and Applications*, 17(4):1020–1024.

Rohn, J. and Shary, S. P. (2018). Interval matrices: Regularity generates singularity. *Linear Algebra and its Applications*, 540:149–159.

Rump, S. (1997). Theorems of perron-frobenius type for matrices without sign restrictions. *Linear Algebra and its Applications*, 266:1–42.

Rump, S. M. (2003a). On p-matrices. *Linear Algebra and its Applications*, 363:237–250.

Rump, S. M. (2003b). Perron–frobenius theory for complex matrices. *Linear Algebra and its Applications*, 363:251–273. Special Issue on Nonnegative matrices, M-matrices and their generalizations.

Takeuchi, Y. (1996). *Global Dynamical Properties of Lotka-Volterra Systems*. World Scientific.

Takeuchi, Y., Adachi, N., and Tokumaru, H. (1978). The stability of generalized volterra equations. *Journal of Mathematical Analysis and Applications*, 62(3):453–473.

Tsatsomeros, M. and Li, L. (2000). A recursive test for p-matrices. *BIT*, 40:410–414.

Wigner, E. P. (1958). On the distribution of the roots of certain symmetric matrices. *Annals of Mathematics*, 67(2):325–327.